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Compact Multi-Retractions and Shape Theory

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Recently Suszycki [6], [7] defined the concept of multi-retractions of compact metric spaces and discussed some properties. In this paper we shall extend that concept to metric spaces and consider some properties related to shape theory.

1. Definitions. In this paper we assume that all spaces are metrizable and all maps are continuous. By a <u>multi-valued function</u> φ from a space X to a space Y we mean a function φ assigning for each point $x \in X$ to a non-empty closed subset $\varphi(x)$ of Y and write $\varphi : X \longrightarrow Y$. In particular if $\varphi(x)$ is compact for every $x \in X$, then we call φ a <u>compact multi-valued function</u>. A multi-valued function $\varphi : X \longrightarrow Y$ is said to be <u>upper semi-continuous</u> (u.s.c.) provided for every point $x \in X$ and every neighborhood V of $\varphi(x)$ in Y there exists a neighborhood U of x in X such that $\varphi(U) = \bigcup_{z \in U} \varphi(z) \subset V$. For a multi-valued function $\varphi : X \longrightarrow Y$ we shall define the <u>graph</u> of φ as follows

$$\Phi = \{(x,y) \in X \times Y \mid y \in \varphi(x),\ x \in X\} .$$

Then let $p : \Phi \longrightarrow X$ and $q : \Phi \longrightarrow Y$ be natural projections. Throughout this paper we shall use these notation.

If a multi-valued function $\varphi : X \longrightarrow Y$ is u.s.c., then the graph Φ of φ is closed in $X \times Y$. Moreover if φ is compact, then the projection $p : \Phi \longrightarrow X$ is proper.

An u.s.c. compact multi-valued function $\varphi : X \longrightarrow Y$ is said to be a <u>compact multi-map</u> (c-multi-map) provided $\varphi(x)$ has the trivial shape for each $x \in X$.

Let Y be a subset of a space X. Then a c-multi-map $\varphi : X \longrightarrow Y$ is said to be a <u>compact multi-retraction</u> (c-multi-retraction) if $y \in \varphi(y)$ for every $y \in Y$.

Let Y be a subset of a space M. If there exist a neighborhood U of Y in M and c-multi-retraction from U to Y, then we call Y a <u>neighborhood compact-multi-retract</u> of M (neighborhood c-multi-retract). In particular if U = M, then we say that Y is a <u>compact multi-retract</u> of M (c-multi-retract).

Remark. Let Y be a subset of a space X. If Y is a (neighborhood) retract of X, then Y is a (neighborhood) c-multi-retract of X. If Y is an FAR, then Y is a c-multi-retract of X. Hence there are a space X and a subset Y of X such that Y is a c-multi-retract of X but not a retract of X.

2. Compact Multi-Retractios. Throughout this section we assume that Y is a subset of a space X and $\varphi X : \longrightarrow Y$ is a c-multi-retraction from X to Y. Then the natural projection $p : \Phi \longrightarrow X$ is a CE-map. Therefore we obtain the following theorem.

Theorem 2.1. Pro- $\pi_n(Y,y)$ is dominated by pro- $\pi_n(X,x)$ in pro- $\mathcal{G}$ and $\check{\pi}_n(Y,y)$ is dominated by $\check{\pi}_n(X,x)$ in $\mathcal{G}$ for every $n \geq 1$ and $y \in \varphi(x)$, where $\mathcal{G}$ is the category of groups and homomorphisms. Pro-$H_n(Y)$ is dominated by pro-$H_n(X)$ in pro- $\mathcal{G}$ and $\check{H}_n(Y)$ is dominated by $\check{H}_n(X)$ in $\mathcal{G}$ for every $n \geq 1$. Moreover $\check{H}^n(Y)$ is dominated by $\check{H}^n(X)$ in $\mathcal{G}$ for every $n \geq 1$.

Theorem 2.1 induces some corollaries.

Corollary 2.2. If X is $AC^n (n \geq 1)$, then so is Y. And if X is acyclic, then so is Y.

Corollary 2.3. If X is compact, connected and pointed S^n-movable$(n \geq 1)$, then so is Y (see [3]).

In particular if X is a pointed 1-movable continuum, then so is Y. Namely c-multi-retractions between continua preserve the pointed 1-movability.

Corollary 2.4. If pro- $\pi_n(X,x)$, $n \geq 1$ and $x \in X$, is stable in pro-$\mathcal{G}$, then pro- $\pi_n(Y,y)$, $y \in \varphi(x)$, is also stable in pro-$\mathcal{G}$.

Then readers may consider that following questions are true.

Question 1. If X is an MAR (resp. MANR), then is Y also an MAR (resp. MANR) ?

Question 2. If X is a pointed movable continuum, then is Y also a pointed movable continuum ?

Question 3. Is it true that $Sd(Y) \leq Sd(X)$, where for a space Z $Sd(Z) = \min \{\dim W \mid Sh(Z) \leq Sh(W)\}$?

Example. Let $f: Y \longrightarrow Q$ be the Taylor's CE-map which does not induce a shape equivalence [8]. Then we define X as the mapping cylinder $(Y \times [0,1] \cup Q)/\sim$ of f, where $\sim$ identifies $(x,1)$ with $f(x)$ for each $x \in X$, and Y is idetified with $Y \times \{0\}$ in X. Moreover a c-multi-retraction $\varphi: X \longrightarrow Y$ is defined as follows

$\varphi([y,t]) = \{y\}$ for every $y \in Y$ and $t \in [0,1)$

$\varphi([z]) = f^{-1}(z)$ for every $z \in Q$.

Then X is homotopy equivalent to Q. Hence X is an FAR. But Y is non-movable and $Sd(X) = +\infty$. Namely Questions 1 - 3 are not true. Then, of course, $Sh(X) \not\geq Sh(Y)$.

Related to Questions 1 - 3 we have some partial positive answers.

Theorem 2.5. If dimX is finite, then $Sh(X) \geq Sh(Y)$.

The proof of Theorem 2.5 is essentially due to Kodama [2].

Corollary 2.6. If dimX is finite, then Questions 1 - 3 are all true.

Remark. In Theorem 2.5 and Corollary 2.6 the assumption of the finite-dimensionality of X is essential by the above example.

In the case of compacta we obtain other answers.

Corollary 2.7. Let $X \supset Y$ be compacta. If X is an FAR and Y is either movable or $Sd(Y) < +\infty$, then Y is an FAR (see [4]).

<u>Corollary</u> 2.8. Let $X \supset Y$ be continua. If X is an FANR and $Sd(Y) < +\infty$, then Y is an FANR (see [1] or [9]).

<u>Corollary</u> 2.9. Let $X \supset Y$ be continua. If X and Y satisfy following conditions

(1) $X \in AC^1$,

(2) X is either movable or $Sd(X) < +\infty$,

(3) Y is either movable or $Sd(Y) < +\infty$,

then $Sd(Y) \leq Sd(X)$ (see [5]).

3. Absolute Neighborhood Compact-Mluti-Retract and Absolute Compact-Multi-Retract. A space Y is said to be an <u>absolute neighborhood compact-multi-retract</u> (m_c-ANR) provided for every space N containing Y as a closed subset Y is a neighborhood c-multi-retract of N. A space Y is said to be an <u>absolute compact-multi-retract</u> (m_c-AR) provided for every space N containing Y as a closed subset Y is a c-multi-retract of N.

By definitions following basic properties of m_c-AR and m_c-ANR are held.

3.1. If Y is an m_c-AR and $Y \underset{\sim}{\Delta} Z$, then Z is also an m_c-AR.

3.2. If Y is an m_c-ANR and $Y \underset{\sim}{\Delta} Z$, then Z is also an m_c-ANR.

3.3. A space Y is an m_c-AR if and only if $Y \subset N \in AR$ as a closed subset is a c-multi-retract of N.

3.4. A space Y is an m_c-ANR if and only if $Y \subset N \in AR$ as a closed subset is a neighborhood c-multi-retract of N.

3.5. A space Y is an m_c-AR if and only if for every closed subset X of a space M and every map $f: X \longrightarrow Y$ there exist a c-multi-map $\varphi: M \longrightarrow Y$ such that $f(x) \in \varphi(x)$ for every $x \in X$.

3.6. A space Y is an m_c-ANR if and only if for every closed subset X of a space M and every map $f: X \longrightarrow Y$ there exist a neighborhood U of X and a c-multi-map $\varphi: U \longrightarrow Y$ such that $f(x) \in \varphi(x)$ for every $x \in X$.

Remarks. 1. If Y is an AR (resp. ANR), then Y is an m_c-AR (resp. m_c-ANR).

2. If Y is an FAR, then Y is an m_c-AR. But there exists a planar 1-dimensional FANR which is not an m_c-ANR (see [7]).

The next problem is still open.

Problem 1. Is it true that every MAR is an m_c-AR ?

Corresponding to results of section 2 we obtain following properties of m_c-AR and m_c-ANR.

3.7. If Y is am m_c-AR, $Y \in AC^{\infty}$, pro-$H_n(Y) = 0$ in pro-$\mathcal{G}$ and $\check{H}_n(Y) = \check{H}^n(Y) = 0$ in $\mathcal{G}$ for every $n \geq 0$.

3.8. If Y is an m_c-ANR, then both pro-$\pi_n(Y,y)$ and pro-$H_n(Y)$ are stable in pro-$\mathcal{G}$ for every $n \geq 1$ and $y \in Y$. Moreover if Y is compact, $\check{\pi}_n(Y,y)$ is countable for every $n \geq 1$ and $y \in Y$ and both $\check{H}_*(Y)$ and $\check{H}^*(Y)$ are finitely generated.

3.9. Every compact connected m_c-ANR is pointed S^n-movable for every $n \geq 1$.

3.10. Let Y be a compact m_c-AR. If Y is either movable or $Sd(Y) < +\infty$, then Y is an FAR. In particular a compactum Y with $Sd(Y) < +\infty$ is m_c-AR if and only if Y is an FAR.

3.11. Every compact m_c-ANR space Y with $Sd(Y) < +\infty$ is an FANR. Moreover if Y is AC^1, then Y has the shape of a finite polyhedron.

3.12. Every finite-dimensional m_c-ANR is an MANR.

Related to properties of m_c-AR and m_c-ANR we have following open problems (c.f. [7]).

Problem 2. Does every compact m_c-ANR space Y with Sd(Y) $< +\infty$ have a shape of a finite polyhedron ?

Problem 3. Is it true that every m_c-ANR space Y with $Sd(Y) < +\infty$ is an MANR ?

Problem 4. Let $g:Y \longrightarrow X$ be a CE-map. Is it true that Y is an m_c-ANR if and only if X is an m_c-ANR ?

Then we shall consider Problem 4.

Lemma 3.13. Let $\varphi:X \longrightarrow Y$ be a multi-map and $g:Y \longrightarrow X$ be a map such that $y \in \varphi(g(y))$ for every $y \in Y$. Then if X is an ANR (resp. AR), Y is an m_c-ANR (resp. m_c-AR).

Corollary 3.14. Let $g:Y \longrightarrow X$ be a CE-map. Then if X is an ANR (resp. AR), Y is an m_c-ANR (resp. m_c-AR).

Remark. Let $f:Y \longrightarrow Q$ be the Taylor's CE-map ([8]). Then by Corollary 3.14 Y is an m_c-AR. But Y is not movable. Then the assumption for Y in properties 3.10 - 3.12 are essential.

Theorem 3.15. Let $g:Y \longrightarrow X$ be a CE-map. Let N be an AR containing X as a closed subset. If there are a neighborhood V of X in N and a c-multi-retraction $\varphi:V \longrightarrow X$ such that $\dim \varphi(z) < +\infty$ for every $z \in V$, then Y is an m_c-ANR. Moreover if V = N, then Y is an m_c-AR.

Corollary 3.16. Let $g:Y \longrightarrow X$ be a CE-map. If X is finite-dimensional and an m_c-ANR (resp. m_c-AR), then Y is an m_c-ANR (resp. m_c-AR).

Remark. In the proof of Theorem 3.15 we essentially use the fact that

$$\mathrm{Sh}(f^{-1}(\varphi(z))) = \mathrm{Sh}(\varphi(z)) \quad \text{for every } z \in V.$$

Then by the similar way we obtain the following.

Theorem 3.15'. Let $g:Y \longrightarrow X$ be a hereditary shape equivalence. If X is an m_c-AR (resp. m_c-ANR), then so is Y.

In fact Corollary 3.16 is the special case of Theorem 3.15'. Then we shall give another problem.

Problem 4'. Let $g:Y \longrightarrow X$ be a hereditary shape equivalence. Then is it true that if Y is an m_c-AR (resp. m_c-ANR), So is Y ?

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